On the hydrodynamic equations for flow of liquid helium II with mutual friction

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The flow of liquid helium II at subcritical velocities can be described by hydrodynamic equations which assume the possibility of independent motions of the two fluid components, but difficulties arise in supercritical flows with forces of mutual friction between the component fluids. The extensive evidence that mutual friction is caused by scattering of the thermal excitations in the velocity fields of the quantized vortex lines suggests that the equations should include mutual forces which are large near the vortex-lines and negligible elsewhere. In their original theory, Hall & Vinen (1956) did assume this, and here the implications of the assumption are examined in detail. Serious difficulties are found in reconciling localization of the mutual friction with fully developed flow in channels, and it is suggested that, although the mutual friction arises from scattering near the vortex lines, the force on the superfluid is distributed uniformly over distances comparable with the distance between adjacent vortex lines.

1. Introduction

The remarkable flow properties of liquid helium II can be described consistently using the two-fluid model of Tisza and London which assumes that a component with Newtonian viscosity—the normal fluid—co-exists with and interpenetrates freely an inviscid component—the superfluid. The detailed interpretation of the model is that the thermal excitations of the liquid are sufficiently localized in space to be regarded as 'molecules' which carry with them energy and momentum in much the same way as do gas molecules and which can acquire drift velocities independent of the velocity of the liquid in which they are propagated. In the two-fluid model, the energy and drift momentum of the excitations are the energy and momentum of the normal fluid and the residual properties of the liquid are the properties of the superfluid, essentially those of liquid helium at absolute zero of temperature. So long as the relative velocities of the two components are small compared with the propagation velocities of the excitations independent motion is possible, but experiment shows that this is true only if the flow velocity is less than a much smaller critical velocity characteristic of the flow system. At higher, supercritical, velocities, momentum interchange between the components takes place, equivalent to forces of mutual friction, which is explained by Hall & Vinen (1956) as a consequence of the scattering of thermal excitations in the strong velocity fields surrounding the vortex lines that appear in the superfluid. The existence of discrete vortex lines with circulation h/m has some theoretical support

(Feynman 1955) and the Hall–Vinen theory of mutual friction has the merit of not requiring additional phenomenonological assumptions. Although the theory has been successful in describing mutual friction in a number of flows, it does not follow that the detailed assumptions about the flow near the vortex lines are either necessary or self-consistent. Indeed, most of the experimental measurements that can be compared with the theory are of flows with line separations small compared with the channel width, and then the existence of the dissipative mechanism of scattering is sufficient to account for the observed mutual friction. If the line separation is comparable with the channel width, the details of the flow become important for the bulk behaviour and cannot be accepted without some discussion of their meaning and consistency with experiment and with the nature of liquid helium. My purpose here is to discuss the hydrodynamic implications of the theory and to show that the original form is not compatible with steady, fully developed flow.

2. The two-fluid equations of motion with mutual friction

The use of hydrodynamic equations of motion to describe flow of liquid helium II approximates the behaviour of an assembly of atoms by that of a continuous fluid and the quantum properties by having two fluid components and quantization of circulation in the superfluid. London (1954) and others have written down equations of this kind, but it may be useful to repeat here the reasoning that leads to the usual forms. Both compressibility and thermal expansion of the whole fluid may be neglected if the flow velocities are small compared with the velocity of ordinary sound and if the temperature differences are small compared with the absolute temperature. With a unit of mass such that the total fluid density is one, ρ_s and ρ_n are the mass fractions of the superfluid and normal fluid components and

$$\rho_s + \rho_n = 1 \quad (\rho_s = \text{function of } T),$$

where T is the absolute temperature. Conservation of mass is expressed by the equation,

$$\operatorname{div}(\rho_s \mathbf{u}_s) + \operatorname{div}(\rho_n \mathbf{u}_n) = 0, \tag{2.1}$$

where $\mathbf{u}_s$ is the superfluid velocity and $\mathbf{u}_n$ is the normal fluid velocity. The local rate of conversion of normal to superfluid is

$$\partial\rho_s/\partial t + \operatorname{div}(\rho_s \mathbf{u}_s) = -\partial\rho_n/\partial t - \operatorname{div}(\rho_n \mathbf{u}_n). \tag{2.2}$$

Conservation of momentum for each fluid is expressed by the equations:

$$\partial(\rho_s \mathbf{u}_s)/\partial t + \operatorname{div}[(\rho_s \mathbf{u}_s)\mathbf{u}_s] = [\partial\rho_s/\partial t + \operatorname{div}(\rho_s \mathbf{u}_s)]\mathbf{u}_s - \rho_s \operatorname{grad} P + \mathbf{F}_{sn}, \tag{2.3}$$

$$\partial(\rho_n \mathbf{u}_n)/\partial t + \operatorname{div}[(\rho_s \mathbf{u}_n)\mathbf{u}_n] = [\partial\rho_n/\partial t + \operatorname{div}(\rho_n \mathbf{u}_n)]\mathbf{u}_s - \rho_n \operatorname{grad} P + \mathbf{F}_{ns} + \nu_n \nabla^2 \mathbf{u}_n, \tag{2.4}$$

where P is the pressure, ν_n is the viscosity of the normal fluid, and $\mathbf{F}_{ns}$, $\mathbf{F}_{sn}$ are mutual forces acting respectively on the normal and the superfluid. To obtain the terms, $[\partial\rho_s/\partial t + \operatorname{div}(\rho_s \mathbf{u}_s)]\mathbf{u}_s$ and $[\partial\rho_n/\partial t + \operatorname{div}(\rho_n \mathbf{u}_n)]\mathbf{u}_s$, which take account of momentum changes by conversion of one fluid to the other, it has been assumed that the conversion takes place at fluid velocity $\mathbf{u}_s$ (London 1954). This assumption simplifies the following argument but is not essential to it.

It is usual to assume that the mutual forces are short-range forces so that, to the approximation implied by the continuum representation, their sum, $\mathbf{F}_{sn} + \mathbf{F}_{ns}$, is everywhere zero. For subcritical flow without mutual friction, their magnitude is determined by two assumptions, based on experiment: (i) that, in isothermal conditions, irrotational and inviscid flow of the superfluid can occur independently of the normal fluid; and (ii) that the entropy resides in and is convected by the normal fluid so that

$$\frac{\partial S}{\partial t} + \operatorname{div}(S\mathbf{u}_n) = \frac{\epsilon}{T} + \frac{k_n}{T^2}(\operatorname{grad} T)^2, \tag{2.5}$$

where S is the local entropy, ϵ is the local rate of production of heat by dissipative processes, and k_n is the thermal conductivity of the normal fluid. Then consideration of the conservation of energy and momentum shows that

$$\mathbf{F}_{sn} = \rho_s S \operatorname{grad} T + \tfrac{1}{2}\rho_s \operatorname{grad}[\rho_n(\mathbf{u}_s - \mathbf{u}_n)^2] \tag{2.6}$$

in flows without mutual friction.

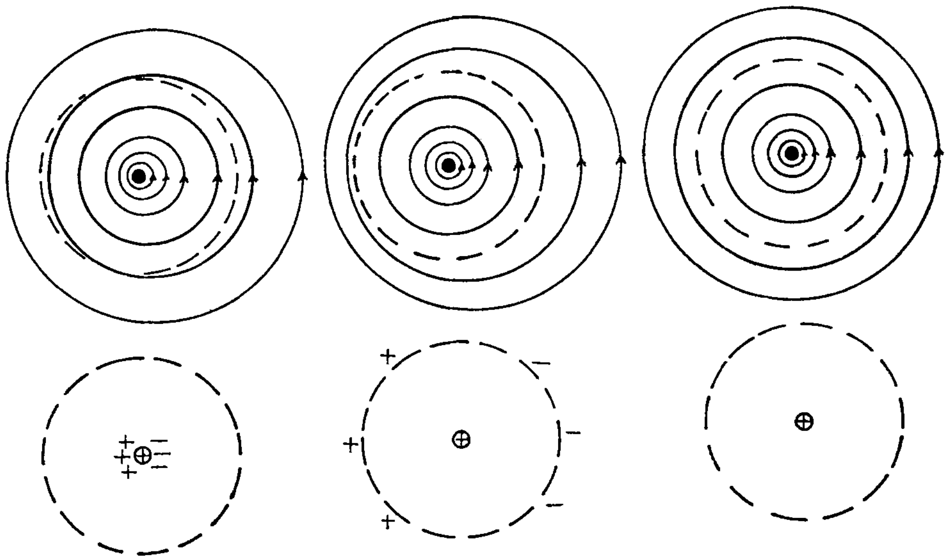
For supercritical flows, terms must be added to these equations to represent the effects of mutual friction. Gorter & Mellink (1949) supposed that the force of mutual friction depends only on the local difference of the velocities of the two components and dimensional reasoning shows the force to be proportional to the cube of the relative velocity (Vinen 1957). The difficulty of accepting a force of this kind is that quantum mechanics requires that the superfluid vorticity should be zero everywhere except possibly on discrete vortex lines. The condition that this condition should persist is easily obtained from equation (2.3) and is that

$$\operatorname{curl}(\mathbf{F}_{sn}/\rho_s) = 0. \tag{2.7}$$

The mutual force of equation (2.6) satisfies this condition but the Gorter–Mellink force of mutual friction does not. Hall & Vinen avoid this particular difficulty by concentrating the forces of mutual friction in the close neighbourhood of the vortex lines, so that non-zero values of curl $\mathbf{u}_s$ occur only in regions so small that the fluid there can no longer be regarded as continuous. The implications and consequences of the vortex-line model of mutual friction are discussed in the next two sections.

3. Structure of vortex lines in motion

In an equilibrium state, such as uniform rotation, the quantized vortex lines move with the superfluid and are simple lines of concentrated, axisymmetrically distributed vorticity. Vorticity and departures from fluid-like behaviour are confined to a core which is believed to have a diameter comparable with the interatomic spacing. The thermal excitations which constitute the normal fluid are propagated in the superfluid and so they are deflected and scattered as they pass through the intense velocity field surrounding each vortex line. If their drift velocity is not the same as the average velocity of the superfluid around the line, a net transfer of momentum takes place between the component fluids at a rate proportional to their velocity difference with a constant of proportionality varying as the vortex-line density. Very briefly, this is the basis of the Hall–Vinen theory of mutual friction which has considerable experimental support (for a full account, see Hall 1960). When the scattering calculations are made, it is found that scattering is appreciable within distances from the core of the order of fifty atomic spacings and so momentum is being lost by the normal fluid inside a cylinder of considerable radius. If this momentum were transferred to the superfluid locally, vorticity would be generated within the scattering volume and the flow there would no longer be irrotational. Confinement of vorticity within the core of the vortex line requires that the forces on the superfluid arising from the scattering process must be transferred either to the core or to the superfluid in general.



(*a*) (*b*) (*c*)

FIGURE 1. Distributions of velocity and vorticity for three kinds of 'discrete' vortex line. (*a*) Vorticity concentrated in a core which moves through the surrounding superfluid (including fluid in the scattering region). (*b*) Core of concentrated vorticity with associated scattering region moving together through the surrounding superfluid. (*c*) Simple core of axisymmetrically distributed vorticity. The full lines are streamlines of velocity relative to the core, the broken circles represent the 'boundary' of the scattering region, and the + and − signs indicate the distribution of vorticity. The velocity of translation of the lines is vertically upwards.

Hall & Vinen chose the first and more natural of these alternatives which means that vortex-doublets are generated continuously at the core and are annihilated continuously by movement of the vortex line through the fluid with velocity **V** in a direction at right-angles to the line and to the total force given by the well-known lift equation

$$\mathbf{F} = \rho_s \mathbf{V} \times \mathbf{K}, \tag{3.1}$$

where **F** is the force per unit length applied to the line, **K** is the vector circulation about the line. In other words, the frictional force is absorbed by adding to the impulse of the system of vortex lines. If vorticity is confined within the line-cores, the average velocity of superfluid in the scattering region is determined by the instantaneous distribution of vorticity in the superfluid and at the flow boundaries and, since the transit velocities of the excitations are much larger than the drift velocities, the scattering depends only on the instantaneous velocity field and is independent of the translation velocity of the line through the fluid given by equation (3.1). The scattering is nearly symmetrical and so the total force must be at right-angles to the line direction and in the plane of the line and the relative velocity between the normal fluid and the superfluid surrounding the scattering region. In their original work, Hall & Vinen chose to assume that the fluid of the scattering region moved with the core through the surrounding fluid and were led to predict a force component at right-angles to the relative velocity. Motion of this kind requires the maintenance of an unquantized distribution of vorticity around the periphery of the scattering region and, since the second force component has not been observed, the assumption seems unnecessary. For the core of a vortex line to move through the superfluid, it is only necessary that the distribution of vorticity within the core should depart slightly from the axisymmetric distribution of an equilibrium system. If the distribution is exactly axisymmetric, the vortex line moves with the fluid. These alternatives are illustrated in figure 1.

4. Fully developed flow in a uniform channel

If liquid flows through a long channel of uniform section, the motion becomes statistically steady and homogeneous in the direction of flow after a sufficient inlet length, and measurements of the dependence of flow-rate on channel length show that this is true for flows of liquid helium II driven either by pressure gradients or by temperature gradients. In the steady state, the total of momentum and impulse of the superfluid is constant and the mean driving force equals the sum of the mean retarding forces, which is the mutual friction if no momentum is transferred to the channel walls. If we accept that the frictional forces act locally on the vortex lines which constitute a very small part of the whole fluid, we should ask how they can balance the sum of the driving force,

$$-\operatorname{grad} P + S \operatorname{grad} T,$$

whose mean value is uniformly distributed over the channel. The difficulty of maintaining such a balance in the steady fully developed flow can be made clear by considering the vorticity of the superfluid. Since the driving-force is the gradient of the scalar,

$$-P + \int_0 S\, dT,$$

its curl is zero and it generates vorticity only at the channel walls. Since the strength of the vortex layer at the walls determines the mean velocity of flow, which is time-independent, the continuous generation of vorticity there must be balanced by absorption of vortex lines of the opposite sign or by emission of vortex lines of the same sign. In an ordinary Newtonian fluid, viscosity diffuses vorticity from the wall but, in superfluid flow, a quantized line close to a wall is moved rapidly by its image in the wall, loses energy and momentum by the forces of mutual friction so generated and very quickly collides with the wall.

For this reason, line emission from the wall is a very improbable process and the generation of wall vorticity can be balanced only by absorption of vortex lines, generated in the flow by stretching of existing lines and then convected or moved to the wall. An immediate difficulty is that motion of lines systematically toward the walls is unlikely unless there is a gradient of vortex-line density across the channel, and thermal flow without net mass-flow is apparently independent of channel width and presumably completely homogeneous.

A more serious difficulty to the concept of forces acting directly on the vortex lines is found by using the expression for the impulse of a fluid containing vortex lines (Lamb 1932, p. 215)

$$\mathbf{P} = \tfrac{1}{2}\int(\operatorname{curl}\mathbf{u}\times\mathbf{x})\,dV. \tag{4.1}$$

If the fluid contains discrete vortex lines with circulation K, the impulse is

$$\mathbf{P} = K\mathbf{A}, \tag{4.2}$$

where $\mathbf{A}$ is the vector area of a surface bounded by the vortex lines. For conservation of momentum, the mean total of the driving force equals the mean total of mutual friction which is the rate of increase of impulse, i.e.

$$A_0\left(-\overline{\frac{\partial P}{\partial x}}+\overline{S\frac{\partial T}{\partial x}}\right) = K\overline{\frac{dA_x}{dt}}, \tag{4.3}$$

where A_0 is the sectional area and A_x is the area of the surface projected on a plane transverse to the flow direction. Consideration of signs shows that vortex-loops of positive area (anti-clockwise circulation when viewed along the direction of superfluid flow) tend to increase in area while loops of negative area tend to decrease, in each case because the velocity of translation of the line in response to the mutual friction is, on the average, in that sense. The distribution of mutual friction along the vortex lines can be considered in two parts, the forces due to the difference of mean velocities of the two fluids and the forces due to fluctuations of the difference. Since the translational velocity of the lines is related to the mutual friction by equation (3.1), the mean-velocity forces displace each element of the projected lines 'outward' at a constant rate, and a short consideration of vortex loops in space will show that the displacements tend to turn over loops of negative projected area so that the projected area becomes positive. Consequently, the total effect is to orientate all the vortex lines and move them to the walls where they may cancel the vorticity generated by the driving forces, but the mean-velocity forces alone cannot generate more lines and the process ends when all the original lines have collided with the walls.

A steady increase of projected area without a continuous loss of line density requires stretching of vortex lines by turbulent movements at a rate at least equal to the loss by collision with the walls, by distorting vortex loops of simple form into coils occupying no more of the channel section but having a larger projected area (figure 2). To coil a vortex loop, the movement of the line in space must be at some stage in the opposite direction to the translation velocity induced by the mean velocity force on the line, and the necessary reversal might be due either to a translation velocity in response to a fluctuation in the relative velocity, $\mathbf{u}_s-\mathbf{u}_n$, or to convection velocity of the superfluid. In either case, it can be shown that differences of superfluid velocity over distances comparable with the mean separation of vortex lines must commonly exceed the mean velocity for coiling to be a probable process. Measurements by Vinen (1957) of mutual friction in heat flow can be analysed to show that the root-mean-square fluctuations of superfluid velocity are probably less than 0·3 of the mean velocity (see appendix), and so the large fluctuations necessary for coiling must be very rare. The

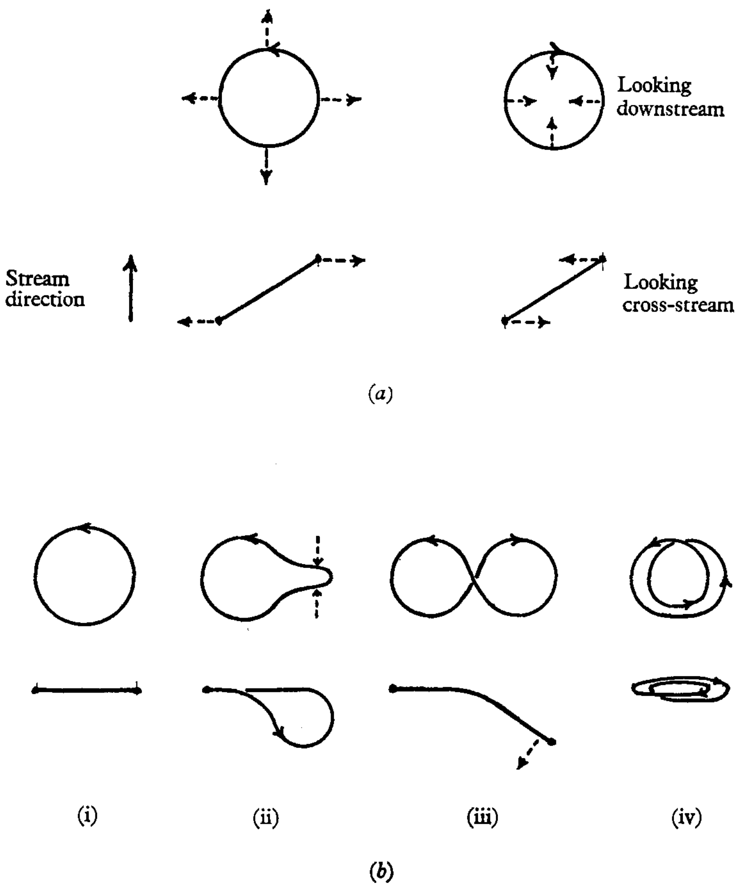


FIGURE 2. Motion of vortex loops. (*a*) Motion of loops with positive and negative projected area in response to 'mean-velocity' frictional forces. (*b*) Stages in the coiling of a vortex loop: (i) initial loop, (ii) twisting of the loop, (iii) folding of the twisted loop, (iv) coiled loop of increased projected area. The full lines represent the vortex cores and the broken arrows their directions of motion. The superfluid velocity is vertical in the cross-stream projection.

difficulty is even greater in isothermal flow in small pressure gradients. There the vortex lines are so few that fluctuations are certainly much less than the mean and, if the vortex lines move in accordance with equation (3.1), they are rapidly removed from the flow.

The conclusion that extension of vortex lines by stretching is unlikely to balance the loss by annihilation at the walls appears at variance with the intense generation of vorticity that occurs in ordinary turbulent flows, but it is a consequence of the systematic motion of the vortex lines in response to the mean velocity force. In an ordinary liquid in turbulent flow, movement and stretching of the vortex lines is controlled by the locally isotropic, small-scale components of the motion and is virtually independent of the mean flow and ordered parts of the motion controlled by the mean flow. If the mutual friction acts directly on the vortex lines of the superfluid, expansion of vortex loops in response to the mean flow is so rapid that the random stretching by turbulent movements can balance loss by annihilation only if velocity fluctuations commonly exceed the mean velocities. This is unknown in ordinary liquids and not to be expected in a superfluid.

5. Discussion

Since the peculiar properties of liquid helium II are quantum effects, it may be that its flow cannot be described except by the methods of quantum mechanics, but it is more likely that the large-scale motion, i.e. aspects of the motion that can be represented with adequate accuracy by averages over regions of space and time large compared with lengths and times on the atomic scale, can be described by the methods of classical hydrodynamics if terms are added to the equations of motion to represent quantum effects. The quantum effects appear as the two-fluid model (of which the thermo-mechanical force is a consequence), the absence of vorticity in the superfluid, the existence of quantized vortex lines and the mutual friction. In the absence of an adequate theory of flow based on quantum mechanics, the form of the classical representation can be found only by considerations of self-consistency and conformity with experimental results. The inclusion of quantized vortex lines and mutual friction is a source of doubt and difficulty, although the work of Hall & Vinen has established that the mutual friction arises from scattering of excitations in the velocity fields surrounding the lines. If vorticity remains concentrated in the vortex lines, the force of mutual friction acting on the superfluid, $\mathbf{f}_{sn}$, cannot generate vorticity and it must be distributed in such a way that $\operatorname{curl} \mathbf{f}_{sn}/\rho_s$ is zero except possibly inside the cores of the lines. The argument of the previous section seems to show that the most natural assumption, that the mutual friction acts only on the line cores, cannot be reconciled with the existence of fully developed channel flow and so we must consider the possibility that the force $\mathbf{f}_{sn}$ is distributed over the whole superfluid.

If the force on the superfluid, $\mathbf{f}_{sn}$, acts almost entirely on the irrotational fluid, it does not add to the impulse of the vortex-line system and, like the driving force, is of the form, $\rho_s \operatorname{grad} \phi$, and ϕ is a scalar. The long-range forces which diffuse momentum transfer derived from scattering of excitations must extend at least as far as the distance between neighbouring lines (or the distance to the wall if that is the less) if the effective distribution is of this form. One reason for expecting long-range forces in the superfluid is that the conservation of irrotational flow has been presented as a basic property of the superfluid and that its classical expression should make provision for the destruction of unquantized vorticity and not merely use the absence of viscosity to show that distributed vorticity does not arise in an initially irrotational flow (as for flows obeying equations (2.3), (2.6)).† A less abstract reason for believing that vortex lines move with the superfluid and so that the friction force is distributed by long-range forces is that line movement relative to the superfluid prevents the occurrence of the vortex-line equivalent of laminar flow, the steady convection along a channel of a regular array of lines. The necessity for an alternative to turbulent flow is shown by some measurements by Atkins of isothermal flow in wide tubes which show clearly a transition from 'laminar' flow to turbulent flow at a Reynolds number comparable with the critical number for Newtonian fluids (Townsend 1961). Further, the flow characteristics on either side of the transition are described quantitatively by analyses based on these descriptions. Other advantages of allowing the existence of laminar flow are found in the description of frictional flows at velocities only slightly above the critical velocity. As has been pointed out before, there is no possibility, let alone likelihood, that random coiling can generate sufficient length of vortex line to balance the loss to the channel walls, but a more positive reason for preferring distributed mutual friction is that it offers a possible explanation of the critical velocity. Consider isothermal flow with a single vortex loop moving along the channel at a velocity set by the general velocity of the superfluid and by its own shape and the wall effects. With a pressure difference between the ends of the channel, a configuration can be found such that the loop moves steadily along the channel and momentum is conserved. The configuration is unstable to small displacements if the general flow velocity is small, but possibly some configurations become stable above a critical velocity. Above this velocity, vortex loops entering the tube could pass through it without risk of destruction. If a continuous supply of loops is available, perhaps generated at slight surface irregularities, mutual friction would appear at the velocity for which convection of isolated vortex loops first becomes a stable process.

The way in which distributed mutual friction should be introduced into the equations of motion for the superfluid is not immediately evident, and in some kinds of flow (e.g. an isolated vortex ring) distribution of the forces may be impossible. My contention is that steady channel flow cannot be described by equations of motion with mutual friction concentrated on the vortex lines. In most of the applications of the vortex-line theory of mutual friction, the mean spacing of the lines is small compared with the flow width and then the necessary requirements of over-all conservation of momentum and energy are sufficient to obtain the answers and the details of the motion are not of direct importance.

Prof. H. E. Hall and Prof. W. F. Vinen have helped me considerably by criticism of earlier forms of my argument and I am grateful to them.

Appendix: The stretching of vortex lines by turbulent movements

If the forces of mutual friction are concentrated at the vortex lines, the motion of the lines consists of an organized 'outward' translation in response to the frictional forces induced by the mean relative velocity, random translation

† The viscous term in the equations of motion can be written $\nu\,\text{curl}\,(\text{curl}\,\mathbf{u})$, and so long-range forces which destroy vorticity also destroy the influence of viscosity.

through the superfluid induced by fluctuations of relative velocity, and random convection by the surrounding superfluid. As pointed out above, the organized translation tends to produce ever-expanding loops of positive area and so to reduce continuously the line density, a decrease that might be counteracted by deformation of existing lines into small loops by movements resembling those of ordinary turbulence. Even if the analogy with ordinary turbulence did not indicate that loop production is most efficient for diameters comparable with the average spacing between lines, it would be clear that the organized expansion would remove all loops of size less than the optimum for production and so we need consider only whether the relative motion of adjacent lines is sufficient to coil them into loops.

The force per unit length on a quantized line of the superfluid is $B\rho_s\rho_n K(u_s-u_n)$, where B is a constant characteristic of the scattering process. If the force is concentrated at the line core, the line is translated through the fluid with velocity V set by equation (3.1). Assuming the relative velocity to be at right-angles to the line, we find

$$V = B\rho_n(u_s-u_n). \tag{A. 1}$$

In thermal flow with no mean mass transfer,

$$\rho_s U_s + \rho_n U_n = 0,$$

where U_s, U_n are the mean velocities of the two fluids, and so

$$\overline{V} = BU_s. \tag{A. 2}$$

The fluctuations in V due to changes in relative velocity depend on the correlation between fluctuations of velocity in the two components. An extreme assumption would be that the fluctuations in mass transfer are zero and then the root-mean-square of the fluctuations of V would be

$$(\overline{v^2})^{\frac{1}{2}} = B(\overline{u_s^2})^{\frac{1}{2}}, \tag{A. 3}$$

where u_s is the fluctuation in superfluid velocity. Viscous forces on the normal fluid would prevent fluctuations as large as this. For coiling to occur, it is necessary that the combined motion of random translation and convection should be capable of twisting a vortex loop in opposition to the systematic expansion with velocity $\overline{V}$. The necessary condition is clearly that the combined random motion should produce relative velocities between adjacent lines that are frequently more than $\overline{V}$.

The difference of convection velocities at adjacent lines depends on the mean spacing and directional correlation of the vortex lines. An approximate estimate of the dependence can be obtained by elementary reasoning. If the total length of vortex line per unit volume is L, the mean spacing is $\frac{1}{2}\sqrt{3}\,L^{-\frac{1}{2}}$ for hexagonal packing ($L^{-\frac{1}{2}}$ for square packing) and, if neighbouring lines are randomly directed (the most favourable configuration), the root-mean-square velocity of a line due to its neighbours is $(3/2\sqrt{2\pi})KL^{\frac{1}{2}}$. The relative velocity is substantially unaffected by more remote lines and it is likely that the R.M.S. relative velocity of convection is no more than $(3/2\sqrt{2\pi})KL^{\frac{1}{2}}$. Measurements by Vinen (1957) show that the mutual friction per unit volume in thermal flow is nearly $A'\rho_s\rho_n(U_s-U_n)^3$, where A' is a constant dependent on temperature. Equating this expression for the mutual friction to the total force on a length L of vortex-line, i.e.

$$B\rho_s\rho_n KL(U_s - U_n) = A'\rho_s\rho_n(U_s - U_n)^3,$$

the ratio of the R.M.S. relative velocity to $\overline{V}$ is

$$\frac{3}{2\sqrt{2}\pi}\frac{KL^{\frac{1}{2}}}{\overline{V}} = \frac{3}{2\sqrt{2}\pi}\left[\frac{A'K}{B}\right]^{\frac{1}{2}}\frac{1}{B\rho_n}.$$

Near 1·4 °K, typical values are $A' = 5{\cdot}6\,\text{cm}^{-2}\,\text{sec}$, $B = 0{\cdot}8$, $K = 10^{-3}\,\text{cm}^2\,\text{sec}^{-1}$, $\rho_n = 0{\cdot}1$, and the ratio is 0·35. Since most of the approximations over-estimate the fluctuations, it is extremely improbable that coiling can occur in the presence of the organized movement of vortex lines.

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